JOURNAL OF FUNCTIONAL ANALYSIS **159**, 203–216 (1998)
ARTICLE NO. FU983289

Hypoellipticity of Some Degenerate Subelliptic Operators

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Received January 20, 1998; accepted March 1, 1998

The purpose of this paper is to establish the following result. Let L_1 and L_2 be subelliptic operators on $\mathbb{R}^n_x$ and $\mathbb{R}^m_y$, respectively. Assume that $\lambda \in C^\infty(\mathbb{R}^n_x)$ with $\lambda \geqslant 0$, assume that λ has a zero of infinite order at the origin and that all other zeroes of λ are of finite order. Then the operator $L = L_1 + \lambda L_2$ is hypoelliptic.
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INTRODUCTION

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$$L_1 = -\sum_1^n a^{ij}(x) \frac{\partial^2}{\partial x_i \, \partial x_j} + \sum_1^n a^i(x) \frac{\partial^2}{\partial x_i} + a(x)$$

$$L_2 = -\sum_1^m b^{ij}(y) \frac{\partial^2}{\partial y_i \, \partial y_j} + \sum_1^m b^i(y) \frac{\partial}{\partial y_i} + b(y)$$

with $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_m)$ where a^{ij}, a^i, $a \in C^\infty(\mathbb{R}^n_x)$ and b^{ij}, b^i, $b \in C^\infty(\mathbb{R}^m_y)$.

The purpose of this paper is to prove the following result.

Main Theorem. *Let L be the differential operator on $\mathbb{R}^n_x \times \mathbb{R}^m_y$ defined by $L = L_1 + \lambda(x) L_2$ with $\lambda \in C^\infty(\mathbb{R}^n_x)$. Assume that $\lambda \geqslant 0$, that λ has a zero of infinite order at the origin, and that all other zeros of λ are of finite order. Then L is hypoelliptic. More precisely, if u is a distribution on $\mathbb{R}^n_x \times \mathbb{R}^n_y$ such that $\zeta L u \in H^s(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ for all $\zeta \in C_0^\infty(U)$ where U is an open set in $\mathbb{R}^n_x \times \mathbb{R}^m_y$, then $\zeta u \in H^s(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ for all $\zeta \in C_0^\infty(U)$. Here $H^s(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ denotes the Sobolev space of functions with s derivatives on $L_2(\mathbb{R}^n_x \times \mathbb{R}^m_y)$.*

Fedii (see [F]) proves hypoellipticity of the operator $-(\partial^2/\partial x^2) - \varphi(x)^2 \, (\partial^2/\partial y^2)$ on $\mathbb{R}^2$, with $\varphi \in C^\infty(\mathbb{R}_x)$, $\varphi(0) = 0$, and $\varphi(x) > 0$ when $x \neq 0$. The result presented here is a generalization to operators, which are not necessarily sums of squares of first order operators and which, outside the degenerate set, are subelliptic. The question of hypoellipticity of operators which are degenerate elliptic has been studied by a number of authors (see, for example, Bell and Mohamed [BM], Christ [Ch], Kusuoka and Stroock [KS], and Morimoto [M]). This work has been motivated by the

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study of regularity properties on pseudoconvex domains, as explained below.

The main result here depends on establishing an estimate of the form

$$\|\zeta u\|_s \leqslant C(\|\zeta' Lu\|_s + \|\zeta' u\|_{-s_0}),$$

where ζ and ζ' are compactly supported cutoff functions with $\zeta' = 1$ in a neighborhood of supp(ζ). The difficulty lies in the fact that the usual estimates would lead to an error term of the form "small constant" times $\|D(\zeta)\,u\|_s$, where D represents a differential operator; such a term cannot be absorbed in the left hand side. The difficulty is overcome by choosing ζ to be a product in such a way that the derivatives of one factor are supported in a region where stronger estimates hold, and the derivatives of the other factor appear in a form that can be estimated inductively. Once the a priori estimate is established, we must use smoothing operators on the actual solution u. The terms arising from commutators with smoothing operators present similar difficulties which are overcome by using "partially" smoothing operators.

The motivation for this work comes from studying the local regularity of the operators $\bar\partial$ and $\bar\partial_b$ on pseudoconvex domains and pseudoconvex $C\mathbb{R}$ manifolds. Local regularity is well understood at points of finite D'Angelo type (see [C] and [D'A]). At points of infinite type, I have analyzed local regularity in the case when the boundary of Ω near the origin has the form

$$\mathrm{Re}(z_n) = \sum_j^N |h_j(z_1, ..., z_{n-1})|^2\, e^{-1/(|z_1|^2 + \cdots + |z_{n-1}|^2)},$$

when the h_j are holomorphic functions in $\mathbb{C}^{n-1}$ with an isolated zero at the origin. The analysis of this example is based on the techniques developed in [K1], [K2], and [K3] as well as those in this note. This work as well as related conjectures is described in [K4].

1. SUBELLIPTICITY

DEFINITION 1.1. Let L be the operator defined by

$$Lu = -\sum a^{ij} u_{x_i x_j} + \sum a^i u_{x_i} + au, \tag{1.1}$$

where a^{ij}, a^i, $a \in C^\infty(\mathbb{R}^n)$ and where $(a^{ij}) \geqslant 0$. Then L is *subelliptic at* $x^0 \in \mathbb{R}^n$ if there exists a neighborhood U of x^0 and positive constants ε and C such that

$$\|u\|_\varepsilon^2 \leqslant C(|(Lu, u)| + \|u\|^2) \tag{1.2}$$

for all $u \in C_0^\infty(U)$. L is called *subelliptic* if it is subelliptic at each point of $\mathbb{R}^n$.

The hypoellipticity of subelliptic operators is given by the following result (see [KN]).

THEOREM 1.2. *If L is subelliptic, then L is hypoelliptic. More precisely: if L is subelliptic at x^0, as above, and if for every $\zeta \in C_0^\infty(V)$ we have $\zeta L(u) \in H^s(\mathbb{R}^n)$, then $\zeta u \in H^{s+2\varepsilon}(\mathbb{R}^n)$ for all $\zeta \in C_0^\infty(V \cup U)$. Furthermore, given ζ'', $\zeta' \in C_0^\infty(V)$, and $\zeta \in C^\infty(U \cap V)$, such that $\zeta' = 1$ in a neighborhood of* supp(ζ) *and $\zeta'' = 1$ in a neighborhood* supp(ζ'). *Then for any pair of positive numbers s, s_0 there exists a constant C (depending on ζ, ζ', ζ'', s, and s_0) such that whenever $\zeta'' u \in H^{s_0}(\mathbb{R}^n)$ and $\zeta' L u \in H^s(\mathbb{R}^n)$ then $\zeta u \in H^{s+2\varepsilon}$ and*

$$\|\zeta u\|_{s+2\varepsilon} \leqslant C(\|\zeta' L u\|_s + \|\zeta'' u\|_{-s_0}). \tag{1.3}$$

We will denote by $\mathbb{R}_x^n$ and $\mathbb{R}_y^m$ the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ with coordinates $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$, respectively. Let L_1 and L_2 be subelliptic operators on $\mathbb{R}_x^n$ and $\mathbb{R}_y^n$, respectively and let $\lambda \in C^\infty(\mathbb{R}_x^n)$ with $\lambda(x) \geqslant 0$ for all $x \in \mathbb{R}_x^n$. Define L on $\mathbb{R}_x^n \times \mathbb{R}_y^m$ by

$$L = L_1 + \lambda L_2. \tag{1.4}$$

Fefferman and Phong (see [FP]) have characterized subellipticity. The following result is a consequence of that characterization.

PROPOSITION 1.3. *If L_1 and L_2 are subelliptic at x^0 and y^0, respectively, then the operator $L_1 + \lambda L_2$ is subelliptic at (x^0, y^0) if, and only if, λ has a zero of finite order at x^0.*

2. A-PRIORI ESTIMATES

The following lemma will be useful in the derivation of the a-priori estimates of this section.

LEMMA 2.1. *Let P and Q be pseudodifferential operators on $C_0^\infty(\mathbb{R}^N)$ of orders p and q, respectively. Assume that $P - P^*$ and $Q - Q^*$ are of order $p-1$ and $q-1$, respectively. Let ζ, η, $\zeta' \in C_0^\infty(\mathbb{R}^N)$ such that $\eta = 1$ on a neighborhood of* supp(ζ_{x_i}). *Then there exists $C > 0$ such that*

$$|(P\zeta u_{x_i}, Q\zeta u)| \leqslant C(\|\zeta u\|^2_{(p+q)/2} + \|\eta u\|^2_{(p+q)/2}) \tag{2.2}$$

for all $u \in C^\infty(\mathbb{R}^N)$.

Proof. Integrating by parts we have

$$\begin{aligned}(P\zeta u_{x_i}, Q\zeta u) = & -(Q\zeta u, P\zeta u_{x_i}) - (P\zeta_{x_i} u, Q\zeta u) \\ & + \left(\left[P, \frac{\partial}{\partial x_i}\right] \zeta u, Q\zeta u\right) - \left(P\zeta u, \left[\frac{\partial}{\partial x_i}, Q\right] \zeta u\right) \\ & - (P\zeta u, Q\zeta_{x_i} u) + \left(\zeta u, (P^*)\, Q \frac{\partial}{\partial x_i} \zeta u\right) \\ & - (\zeta u, (P - P^*)\, Q\zeta_{x_i} u) - \left(\zeta u, [P, Q] \frac{\partial}{\partial x_i} \zeta u\right) \\ & + (\zeta u, [P, Q]\, \zeta_{x_i} u) + \left((Q - Q^*)\, \zeta u, P \frac{\partial}{\partial x_i} \zeta u\right) \\ & - ((Q - Q^*)\, \zeta u, P\zeta_{x_i} u).\end{aligned}$$

Since $(P\zeta u_{x_i}, Q\zeta u) = (Q\zeta u, P\zeta u_{x_i})$ we have expressed $(P\zeta u_{x_i}, Q\zeta u)$ as a combination of terms of the form $(A\zeta_{x_i} u, B\zeta u)$ and $A\zeta u, B\zeta u)$, where A and B are of orders a and b, respectively, with $a + b \leqslant p + q$. Now we have

$$\begin{aligned}|(A\zeta_{x_i} u, B\zeta u)| &= |(\Lambda^{(b-a)/2} A\zeta_{x_i} \eta u, \Lambda^{(a-b)/2} B\zeta u)| \\ &\leqslant C\, \|\eta u\|_{(a+b)/2}\, \|\zeta u\|_{(a+b)/2} \\ &\leqslant C\, \|\eta u\|_{(p+q)/2}\, \|\zeta u\|_{(p+q)/2}.\end{aligned}$$

Similarly, we have

$$|(A\zeta u, B\zeta u)| \leqslant C\, \|\zeta u\|^2_{(p+q)/2}.$$

Hence (2.2) follows and the lemma is proved.

LEMMA 2.2. *Suppose that L, given by* (1.1), *is subelliptic at x^0 and that $\zeta, \zeta' \in C_0^\infty(U)$, with U a neighborhood of x^0 as in* (1.2), *further suppose that $\zeta' = 1$ in a neighborhood of x^0, then there exists a constant $C > 0$ such that*

$$\|\zeta u\|^2_\varepsilon \leqslant C\{\Sigma(a^{ij} \zeta u_{x_i}, \zeta u_{x_j}) + \|\zeta' u\|^2\} \leqslant C\{|(\zeta L u, \zeta u)| + \|\zeta' u\|^2\}. \quad (2.3)$$

for all $u \in C^\infty(\mathbb{R}^n)$.

Proof. From (1.2) we have

$$\|\zeta u\|^2_\varepsilon \leqslant C(|(L(\zeta u), \zeta u)| + \|\zeta u\|^2)$$

and

$$(L(\zeta u), \zeta u) = -\sum (a^{ij}(\zeta u)_{x_i x_j}, \zeta u) + \left(\sum (a^i(\zeta u)_{x_i} + a\zeta u, \zeta u)\right).$$

By Lemma 2.1 the second term on the right is $0(\|\zeta' u\|^2)$. For the first term, by integration by parts and Lemma 2.1, we have

$$-\Sigma(a^{ij}(\zeta u)_{x_i x_j}, \zeta u) = \Sigma(a^{ij}\zeta u_{x_i}, \zeta u_{x_j}) + 0(\|\zeta' u\|^2).$$

This gives the first part of (2.3). To obtain the second part, we again use integration by parts and Lemma 2.1 to get

$$\Sigma(a^{ij}\zeta u_{x_i}, \zeta u_{x_j}) = (\zeta L u, \zeta u) + 0(\|\zeta' u\|^2),$$

which concludes the proof of (2.3).

Given $u \in C_0^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^m)$ we define the partial Fourier transforms $\mathscr{F}_x u$ and $\mathscr{F}_y u$ by

$$\begin{cases} \mathscr{F}_x u(y, \xi') = \int_{\mathbb{R}_x^m} e^{-ix\cdot\xi'} u(x, y)\, dx \\ \mathscr{F}_y u(x, \xi'') = \int_{\mathbb{R}_n^y} e^{-iy\cdot\xi''} u(x, y)\, dy. \end{cases} \tag{2.4}$$

For $s \in \mathbb{R}$ we define the operators Λ_x^s and Λ_y^s by

$$\begin{cases} \mathscr{F}_x(\Lambda_x^s u)(y, \xi') = (1 + |\xi'|^2)^{s/2}\, \mathscr{F}_x u(\xi', y) \\ \mathscr{F}_y(\Lambda_y^s u)(x, \xi'') = (1 + |\xi''|^2)^{s/2}\, \mathscr{F}_y(x, \xi''). \end{cases} \tag{2.5}$$

LEMMA 2.3. *Suppose that L_1 and L_2 are subelliptic at $x^0 \in \mathbb{R}_x^n$ and $y^0 \in \mathbb{R}_y^m$, respectively. Let $U_1 \subset \mathbb{R}_x^n$ and $U_2 \subset \mathbb{R}_y^m$ be neighborhoods of x^0 and y^0, respectively such that* (1.2) *holds with ε_1 for L_1 on U_1 and with ε_2 for L_2 on U_2. Suppose that $\lambda \in C^\infty(\mathbb{R}_x^n)$ is non-negative. Then if $\zeta, \zeta' \in C_0^\infty(U_1 \times U_2)$, with $\zeta' = 1$ on a neighborhood of* supp(ζ), *we have*

$$\begin{aligned} &\|\Lambda_x^\varepsilon(\zeta u)\|^2 + \|\sqrt{\lambda}\, \Lambda_y^\varepsilon(\zeta u)\|^2 \\ &\quad \leqslant C\left\{\sum_1^m (a^{ij}\zeta u_{x_i}, \zeta u_{x_j}) + \sum_1^m (\lambda b^{ij}\zeta u_{y_i}, \zeta u_{y_j}) + \|\zeta' u\|^2\right\} \\ &\quad \leqslant C\{|(\zeta L u, \zeta u)| + \|\zeta' u\|^2\}. \end{aligned} \tag{2.6}$$

Here

$$\begin{cases} L_1 = -\sum_1^n a^{ij} \dfrac{\partial^2}{\partial x_i \partial x_j} + \sum_1^n a^i \dfrac{\partial}{\partial x_i} + a \\ L_2 = -\sum_1^m b^{ij} \dfrac{\partial^2}{\partial y_i \partial y_j} + b^i \dfrac{\partial}{\partial y_i} + b \\ L = L_1 + \lambda L_1, \end{cases} \tag{2.7}$$

where $a^{ij} a^i$, a, $\lambda \in C^\infty(\mathbb{R}^n_x)$, b^{ij}, b^i, $b \in C^\infty(\mathbb{R}^m_y)$ *and* $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$.

Proof. From (2.13) we conclude that for each $y \in \mathbb{R}^m_y$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n_x} |\Lambda^\varepsilon_x(\zeta u)(x, y)|^2 \, dx \\ &\qquad \leqslant C \left\{ \sum \int_{\mathbb{R}^m_x} a^{ij}(x) \, \zeta(x, y) \, u_{x_i}(x, y) \, \zeta(x, y) \, u_{x_j}(x, y) \, dx \right. \\ &\qquad\qquad \left. + \int_{\mathbb{R}_{x^n}} |\zeta'(x, y) \, u(x, y)|^2 \, dx \right\}, \end{aligned} \tag{2.8}$$

where C is independent of y. Similarly for each $x \in \mathbb{R}^n_x$, we have

$$\begin{aligned} &\int_{\mathbb{R}^n_y} \lambda(x) \, |\Lambda^\varepsilon_x(\zeta u)(x, y)|^2 \, dy \\ &\qquad \leqslant C \left\{ \sum \int_{\mathbb{R}^n_y} \lambda(x) \, b^{ij}(y) \, \zeta(x, y) \, u_{y_i}(x, y) \, \zeta(x, y) \, u_{y_j}(x, y) \, dy \right. \\ &\qquad\qquad \left. + \int_{\mathbb{R}^n_y} |\zeta'(x, y) \, u(x, y)|^2 \, dy \right\}. \end{aligned} \tag{2.9}$$

Now integrating (2.8) with respect to y and (2.9) with respect to x and adding, we obtain the first part of (2.6). The second part then follows by integration by parts and Lemma 2.1 noting that $\partial \lambda / \partial y_j = 0$.

We choose cutoff functions $\sigma, \tilde{\sigma}, \sigma' \in C_0^\infty(\mathbb{R}^n_x)$ and $\theta, \tilde{\theta}, \theta' \in C_0^\infty(\mathbb{R}^m_y)$ such that $\sigma = 1$ in a neighborhood of $0 \in \mathbb{R}^n_x$, $\tilde{\sigma} = 1$ in a neighborhood of $\operatorname{supp}(\sigma)$, $\sigma' = 1$ in a neighborhood of $\operatorname{supp}(\tilde{\sigma})$, and $\tilde{\theta} = 1$ in a neighborhood of $\operatorname{supp}(\theta)$, and $\theta' = 1$ in a neighborhood of $\operatorname{supp}(\tilde{\theta})$. Let $\zeta(x, y) = \sigma(x) \, \theta(y)$, $\tilde{\zeta}(x, y) = \tilde{\sigma}(x) \, \tilde{\theta}(y)$, and $\zeta'(x, y) = \sigma'(x) \, \theta'(y)$. Let U_0 and U be neighborhoods of the origin in $\mathbb{R}^n_x$ such that $\bar{U}_0 \subset U$ and $\sigma = 1$ on U. Let $\sigma_0, \tilde{\sigma}_0 \in C_0^\infty(\mathbb{R}^n_x)$ with $\operatorname{supp}(\sigma_0) \cap U_0 = \phi$, $\sigma_0 = 1$ in a neighborhood of $\bigcup \operatorname{supp}(\sigma_{x_i})$. Set $\zeta_0 = \sigma_0 \tilde{\theta}$. Thus we have $\zeta_0 \zeta_{x_i} = \zeta_{x_i}$.

From now on we assume that if $\lambda(x^0)=0$ and $x^0 \neq 0$, then x_0 is a zero of finite order. Then, for any $y^0 \in \mathbb{R}^m_y$, the operator $L = L_1 + \lambda L_2$ is subelliptic at (x^0, y^0). Thus, in particular, L is subelliptic at (x^0, y^0) whenever $(x^0, y^0) \in \operatorname{supp}(\tilde{\zeta}_0)$. Hence, by (2.3), there exist $C>0$ and $\varepsilon>0$ such that

$$\|\zeta_0 u\|_\varepsilon^2 \leqslant C\{|(\zeta_0 Lu, \zeta_0 u)| + \|\tilde{\zeta} u\|^2\} \tag{2.10}$$

for all $u \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^m_y)$.

We denote the coordinates in $\mathbb{R}^n_{\xi'} \times \mathbb{R}^m_{\xi''}$ by

$$\xi_i = \begin{cases} \xi'_i & \text{for} \quad i=1, ..., n \\ \xi''_{i+n} & \text{for} \quad i+u, ..., m+n \end{cases}.$$

Then the Fourier transform of $u \in C_0^\infty(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ is given by

$$\hat{u}(\xi) = \int_{\mathbb{R}^n_x \times \mathbb{R}'_y\, m} e^{-ix \cdot \xi' - iy \cdot \xi''} u(x, y)\, dx\, dy$$

and for $s \in \mathbb{R}$ we define $\Lambda^s u$ and $\|u\|_s$, by

$$\widehat{\Lambda^s u}(\xi) = (1+|\xi|^2)^{s/2}\, \hat{u}(\xi)$$

and

$$\|u\|_s = \|\Lambda^s u\|.$$

Lemma 2.4. *Given s and the cutoff functions defined above, there exists $C>0$ such that*

$$|(\tilde{\zeta}[L, \Lambda^s \zeta]\, u, \tilde{\zeta} \Lambda^s \zeta u)| \leqslant C\{\|\zeta_0 u\|_s^2 + \|\lambda \Lambda^s \tilde{\zeta} u\|^2 + \|\zeta u\|_s^2 + \|\zeta' u\|_{s-1}^2\} \tag{2.11}$$

for all $u \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^m_y)$.

Proof.

$$\begin{aligned}
[L, \Lambda^s \zeta]\, u &= [L, \Lambda^s]\, \zeta u + \Lambda^s [L, \zeta]\, u \\
&= \sum [a^{ij}, \Lambda^s](\zeta u)_{x_i x_j} + \sum [a^i, \Lambda^s](\zeta u)_{x_i} + [a, \Lambda^s]\, \zeta u \\
&\quad - \sum \lambda [b^{ij} \Lambda^s](\zeta u)_{y_i y_j} + [\lambda, \Lambda^s] \sum b^{ij} (\zeta u)_{y_i y_j} \\
&\quad + \sum \lambda [b^i, \Lambda^s](\zeta u)_{y_i} + [\lambda, \Lambda^s] \sum b^i (\zeta u)_{y_i} + [b, \Lambda^s]\, \zeta u
\end{aligned} \tag{2.12}$$

$$\Lambda^s[L, \zeta]\, u = -\Lambda^s \sum a^{ij}(\zeta_{x_i} u_{x_j} + \zeta_{x_j} u_{x_i}) + \Lambda^s \sum a^i \zeta_{x_i} u - \Lambda^s \sum a^{ij} \zeta_{x_i x_j} u$$

$$= \Lambda^s a \zeta u - \Lambda^s \sum \lambda b^{ij}(\zeta_{y_i} u_{y_j} + \zeta_{y_j} u_{y_i})$$

$$+ \Lambda^s \sum \lambda b^i \zeta_{y_i} u - \Lambda^s \sum \lambda b^{ij} \zeta_{y_i y_j} u + \Lambda^s \lambda b \zeta u \tag{2.13}$$

Since the a^{ij} and λ are independent of y and since the b^{ij} are independent of x then, using the calculus of pseudodifferential operators we obtain

$$\sum [a^{ij}, \Lambda^s] = -s \sum a^{ij}_{x_p} \Lambda^{s-2} + R_1^{s-2}$$

$$\sum [\lambda b^{ij}, \Lambda^s] = \lambda \sum [b^{ij}, \Lambda^s] + [\lambda, \Lambda^s] \sum b^{ij} \tag{2.14}$$

$$= -s\lambda \sum b^{ij}_{y_p} \Lambda^{s-2} \frac{\partial}{\partial y_p} - s \sum \lambda_{x_p} b^{ij} \Lambda^{s-2} \frac{\partial}{\partial x_p} + R_2^{s-2},$$

here R_1^{s-2} and R_2^{s-2} denote pseudodifferential operators of order $s-2$. Setting $P = \tilde{\zeta} \sum a^{ij}_{x_p} \Lambda^{s-2}(\partial^2/\partial x_i\, \partial x_j)$ we have $P - P^*$ is of order $s-1$ since $a^{ij}_{x_p} = a^{ji}_{x_p}$ so we can apply Lemma 2.1 with $Q = \tilde{\zeta} \Lambda^s$ and u replaced by ζu, and we obtain

$$\left| \left(\tilde{\zeta} \sum [a^{ij}, \Lambda^s](\zeta u)_{x_i x_j}, \tilde{\zeta} \Lambda^s \zeta u \right) \right|$$

$$\leqslant C \left\{ \left| \left(\tilde{\zeta} a^{ij}_{x_p} \Lambda^{s-2} \frac{\partial^2}{\partial_{s_i} \partial_{x_j}} (\zeta u)_{x_p}, \tilde{\zeta} \Lambda^s \zeta u \right) \right| + \|\zeta u\|_s^2 \right\}$$

$$\leqslant C' \, \|\zeta u\|_s^2.$$

Similarly we obtain

$$\left| \left(\tilde{\zeta} \sum [\lambda b^{ij} \Lambda^s](\zeta u)_{y_i y_j}, \tilde{\zeta} u \right) \right| \leqslant C \, \|\zeta u\|_s$$

and using Lemma 2.1 on the inner products which arise from the remaining terms in (2.12) we get

$$|(\tilde{\zeta}[L, \Lambda^s]\, \zeta u, \tilde{\zeta} \Lambda^s \zeta u)| \leqslant C \, \|\zeta u\|_s^2. \tag{2.15}$$

Next we deal with (2.13). Denote by Eu the first three terms of (2.13). Since $\zeta_0 = 1$ in a neighborhood of $\bigcup \operatorname{supp}(\zeta_{x_i})$ we have $Eu = E\zeta_0 u$. Thus, applying Lemma 2.1, we have

$$|(\tilde{\zeta} E u, \tilde{\zeta} \Lambda^s \zeta u)| \leqslant C(\|\zeta_0 u\|_s^2 + \|\zeta u\|_s^2). \tag{2.16}$$

Denoting by Fu the remaining terms of (2.13) and noting that $\lambda_{y_i}=0$, we obtain

$$|(\tilde{\zeta}Fu, \tilde{\zeta}\Lambda^s\zeta u)| \leqslant C\left(\|\zeta u\|_s^2 + \|\lambda\Lambda^s\tilde{\zeta}u\|^2 + \sum \|\lambda^{1/2}\Lambda^s\zeta_{y_i}u\|^2 + \|\zeta' u\|_{s-1}^2\right). \quad (2.17)$$

We will use the following well known inequality (see, for instance, [NT] p. 341) which applies to functions $\zeta \geqslant 0$ of compact support

$$\zeta_{y_i}^2 \leqslant C\zeta. \quad (2.18)$$

We have

$$\|\lambda^{1/2}\Lambda^s\zeta_{y_i}u\| \leqslant \|\lambda^{1/2}\zeta_{y_i}\Lambda^s\tilde{\zeta}u\| + \|\lambda^{1/2}[\Lambda^s, \zeta_{y_i}]\,\tilde{\zeta}u\|$$

hence

$$\begin{aligned}\|\lambda^{1/2}\Lambda^s\zeta_{y_i}u\|^2 &\leqslant 2(\lambda\zeta_{y_i}^2\Lambda^s\tilde{\zeta}u, \Lambda^s\tilde{\zeta}u) + C\,\|\zeta' u\|_{s-1}^2 \\ &\leqslant C(\|\lambda\Lambda^s\tilde{\zeta}u\|^2 + \|\zeta\Lambda^s\tilde{\zeta}u\|^2 + \|\zeta' u\|_{s-1}^2) \\ &\leqslant C\{\|\lambda\Lambda^s\tilde{\zeta}u\|^2 + \|\Lambda^s\zeta u\|^2 + \|\zeta' u\|_{s-1}^2\}. \end{aligned} \quad (2.19)$$

Thus (2.11) follows from combining (2.15), (2.16), (2.17), and (2.19). This completes the proof of the lemma.

LEMMA 2.5. *Under the same assumptions as above, we have*

$$\|\Lambda_x^\varepsilon\tilde{\zeta}\Lambda^s\zeta u\|^2 + \|\sqrt{\lambda}\,\Lambda_y^\varepsilon\tilde{\zeta}\Lambda^s\zeta u\|^2 \leqslant C(|\tilde{\zeta}\Lambda^s\zeta Lu, \tilde{\zeta}\Lambda^s\zeta u| + \|\zeta_0 u\|_s^2 + \|\zeta u\|_s^2 + \|\lambda\Lambda^s\tilde{\zeta}u\|^2 + \|\zeta' u\|_{s-1}^2) \quad (2.20)$$

for all $u \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^m)$.

Proof. In (2.6) replace ζ by $\tilde{\zeta}$ and u by $\Lambda^s\zeta u$ to get

$$\begin{aligned}&\|\Lambda_x^\varepsilon\tilde{\zeta}\Lambda^s\zeta u\|^2 + \|\sqrt{\lambda}\,\Lambda_y^\varepsilon\tilde{\zeta}\Lambda^s\zeta u\|^2 \\ &\quad\leqslant C(|\tilde{\zeta}L\Lambda^s\zeta u, \tilde{\zeta}\Lambda^s\zeta u| + \|\Lambda^s\zeta u\|^2) \\ &\quad\leqslant C\{|(\tilde{\zeta}\Lambda^s\zeta Lu, \tilde{\zeta}\Lambda^s\zeta u)| + |(\tilde{\zeta}[L, \Lambda^s\zeta]\,u, [\tilde{\zeta}\Lambda^s\zeta]\,u)| + \|\zeta u\|_s^2\}.\end{aligned}$$

Then applying (2.11) to the penultimate term on the right we get (2.20)

LEMMA 2.6. *There exists $C>0$ such that*

$$\|\zeta_0 u\|_s^2 \leqslant C(|(\zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} Lu, \zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} u)| + \|\zeta' u\|_{s-\varepsilon}^2) \tag{2.21}$$

for $u \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^m)$.

Proof. We start with

$$\|\zeta_0 u\|_s = \|\Lambda^{s-\varepsilon}\zeta_0\tilde{\zeta} u\|_\varepsilon \leqslant \|\zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} u\|_e + c\,\|\zeta' u\|_{s-1}.$$

Now substituting $\Lambda^{s-\varepsilon}\tilde{\zeta} u$ for u in (2.10), we obtain

$$\begin{aligned}\|\zeta_0 u\|_s^2 &\leqslant C\{|(\zeta_0 L\Lambda^{s-\varepsilon}\tilde{\zeta} u, \zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} u)| + \|\zeta' u\|_{s-\varepsilon}^2\} \\ &\leqslant C\{|(\zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} Lu, \zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} u)|\} \\ &\qquad + \{|(\zeta_0 [L, \Lambda^{s-\varepsilon}\tilde{\zeta}]\, u, \zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} u)| + \|\zeta' u\|_{s-\varepsilon}^2\}.\end{aligned}$$

Now substituting ζ_0 for $\tilde{\zeta}$, $s-\varepsilon$ for s, and $\tilde{\zeta}$ for ζ in (2.11) we obtain (2.21).

LEMMA 2.7. *There exists $C>0$ such that*

$$\|\lambda\tilde{\zeta} u\|_s^2 \leqslant C\{|(\tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' Lu, \tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' u)| + \|\zeta' u\|_{s-\varepsilon}^2\} \tag{2.22}$$

for all $u \in C^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^n)$.

Proof. We have

$$\|\lambda\tilde{\zeta} u\|_s^2 = \|\Lambda^s \lambda\tilde{\zeta}\zeta' u\| \leqslant \|\Lambda^\varepsilon \lambda\tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' u\| + C\,\|\zeta' u\|_{s-1}$$

then

$$\|\lambda\tilde{\zeta} u\|_s^2 \leqslant c\{\|\Lambda_x^\varepsilon \lambda\tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' u\|^2 + \|\Lambda_y^\varepsilon \lambda\tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' u\|^2 + \|\zeta' u\|_{s-1}^2\}.$$

Since $[\Lambda_x^\varepsilon, \lambda]$ is bounded and $[\Lambda_y^\varepsilon, \lambda]=0$ we have

$$\|\lambda\tilde{\zeta} u\|_s^2 \leqslant C\{\|\Lambda_x^\varepsilon \tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' u\|^2 + \|\sqrt{\lambda}\,\Lambda_y^\varepsilon \tilde{\zeta}\Lambda^{s-\varepsilon}\zeta u\|^2 + \|\zeta' u\|_{s-\varepsilon}^2\}.$$

Hence substituting $\tilde{\zeta}$ for ζ, $s-\varepsilon$ for s, and ζ' for ζ in (2.20), we obtain (2.22) as desired.

PROPOSITION 2.8. *There exists $C>0$ such that*

$$\begin{aligned}\|\zeta u\|_s^2 \leqslant C\{&|(\tilde{\zeta}\Lambda^s \zeta Lu, \tilde{\zeta}\Lambda^s \zeta u)| + |(\zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} Lu, \zeta_0 \Lambda^{s-\varepsilon}\tilde{\zeta} u)| \\ &+ |(\tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' Lu, \tilde{\zeta}\Lambda^{s-\varepsilon}\zeta' u)| + \|\zeta' u\|_{s-\varepsilon}^2\}\end{aligned} \tag{2.23}$$

and

$$\|\zeta u\|_s \leqslant C(\|\zeta' Lu\|_s + \|\zeta' u\|_{s-\varepsilon}) \tag{2.24}$$

for all $u \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^m_y)$.

Proof. By taking $\tilde{\sigma} \in C_0^\infty(\mathbb{R}^n_x)$ such that the diameter of $\operatorname{supp}(\tilde{\sigma})$ is small, we have

$$\begin{aligned}\|\zeta u\|_s^2 = \|\Lambda^s \tilde{\zeta}\zeta u\|^2 &\leqslant \|\tilde{\zeta}\Lambda^s \zeta u\|^2 + C\,\|\zeta u\|_{s-1}^2\\ &\leqslant \text{small const. } \|\Lambda_x^\varepsilon \tilde{\zeta}\Lambda^s\zeta u\|^2 + C\,\|\zeta u\|_{s-1}^2.\end{aligned}$$

Thus if the "small const." is sufficiently small, the term $\|\zeta u\|_s$ on the right in (2.20) is absorbed in the left hand side, and the estimates (2.23) and (2.24) then follow concluding the proof of the proposition.

3. PARTIALLY SMOOTHING OPERATORS

We are now in a position to prove the principal result of this paper.

MAIN THEOREM 3.1. *The operator* L *is hypoelliptic. More precisely if* $U \subset \mathbb{R}^n_x \times \mathbb{R}^m_y$ *is open, if* u *is a distribution on* $\mathbb{R}^n_x \times \mathbb{R}^m_y$ *and, if* $\zeta Lu \in H^s(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ *for all* $\zeta \in C_0^\infty(U)$, *then* $\zeta u \in H^s(\mathbb{R}^n_x \times \mathbb{R}^m_y)$ *for all* $\zeta \in C_0^\infty(U)$.

To prove this result we will use a "partially smoothing operator" to show that whenever $\zeta Lu \in H^s$ for all $\zeta \in C_0^\infty(U)$, and $\zeta u \in H^{s-\varepsilon}$ for all $\zeta \in C_0^\infty(U)$, then $\zeta u \in H^s$. This will suffice to prove the theorem, since for any distribution u there exists s_0 such that $\zeta u \in H^{-s_0}$ hence we conclude that, if $\zeta Lu \in H^{-s_0+\varepsilon}$, then $\zeta u \in H^{-s_0+\varepsilon}$ and so the result is proved by induction on k, if $\zeta Lu \in H^{-s_0+k\varepsilon}$.

DEFINITION 3.2. For $\delta > 0$ we define S_δ by

$$\widehat{S_\delta u}(\xi) = \frac{\hat{u}(\xi)}{(1+\delta^2\,|\xi|^2)^{3/2}}. \tag{3.1}$$

LEMMA 3.3. *The operator* S_δ *has the following properties.*

(1) *If* $u \in H^s$ *then* $S_\delta u \in H^{s+3}$.

(2) S_δ: $H^s \to H^s$ *are bounded operators with bounds independent of* δ.

(3) *If* $u \in H^{s-3}$ *and* $\|S_\delta u\|_s \leqslant C$ *with* C *independent of* δ *then* $u \in H^s$.

(4) *If* $a \in C_0^\infty(\mathbb{R}_x^n \times \mathbb{R}_y^m)$ *then*

$$[a, S_\delta] = \left(\sum a_{x_j} \frac{\partial}{\partial x_j} + \sum a_{y_i} \frac{\partial}{\partial y_i}\right) R_\delta^{(-2)} S_\delta + R_\delta^{(-1)}, \tag{3.2}$$

where $R_\delta^{(-2)}$ *and* $R_\delta^{(-1)}$ *are families of pseudodifferential operators uniformly of order* -2 *and* -1, *respectively. That is,* $\|R_\delta^{(-2)} u\|_s \leqslant C \|u\|_{s-2}$ *and* $\|R_\delta^{(-1)} u\|_s \leqslant C \|u\|_{s-1}$ *with* C *independent of* δ. *Furthermore,* $R_\delta^{(-2)}$ *is a self adjoint operator on* L_2.

(5) *If* P *is a pseudodifferential operator of order* p, *then* $[P, R_\delta^{(k)}]$ *a pseudodifferential operators of order* $p+k-1$ *uniformly in* δ.

Proof. (1), (2), and (3) follows immediately from the definition of S_δ. To prove (4) we calculate the principal symbol of $[a, S_\delta]$ and obtain principal symbol

$$[a, S_\delta] = \frac{1}{\sqrt{-1}} \left(\sum a_{x_j} \frac{\partial}{\partial \xi'_j} + \sum a_{y_j} \frac{\partial}{\partial \xi''_j}\right)\left(\frac{1}{1+\delta^2 |\xi|^2}\right)^{3/2}$$

$$= \sqrt{-1} \left(\sum a_{x_j} \xi'_j + \sum a_{y_j} \xi''_j\right) \frac{(3/2)\, \delta^2}{(1+\delta^2 |\xi|^2)^{5/2}}.$$

We have

$$\frac{(3/2)\, \delta^2}{(1+\delta^2 |\xi|^2)^{5/2}} = \frac{(3/2)\, \delta^2 |\xi|^2}{(1+\delta^2 |\xi|^2)^{5/2}} \frac{1}{(1+|\xi|^2)} + \frac{(3/2)\, \delta^2}{(1+\delta^2 |\xi|^2)^{5/2}\, (1+|\xi|^2)}. \tag{3.3}$$

Let $R_\delta^{(-2)}$ be the operator defined by

$$(\widehat{R_\delta^{(-2)} u})(\xi) = \frac{(3/2)\, \delta^2 |\xi|^2}{(1+\delta^2 |\xi|^2)} \frac{1}{(1+|\xi|^2)} \hat{u}(\xi).$$

Note that the operator defined by the second term on the right of (3.3) is of order -4 uniformly in δ. Then the difference between $[a, S_\delta]$ and the first term on the right of (3.2) is an operator of order -1 uniformly in δ, which proves (3.2).

Property (5) is an immediate consequence of the formula for commutators of pseudo-differential operators.

Proof of the Main Theorem. To prove the theorem we will show that there exists C, independent of δ, such that

$$\|S_\delta \zeta u\|_s \leqslant C(\|\zeta' Lu\|_s + \|\zeta' u\|_{s-\varepsilon}) \tag{3.4}$$

for all u with $\zeta' Lu \in H^s$ and $\zeta' u \in H^{s-\varepsilon}$.

We start with

$$\begin{aligned}\|S_\delta \zeta u\|_s &\leqslant \|S_\delta \zeta\zeta' u\|_s \leqslant \|\zeta S_\delta \zeta' u\|_s + \|[S_\delta, \zeta]\, \zeta' u\|_s \\ &\leqslant \|\zeta S_\delta \zeta' u\|_s + C\, \|\zeta' u\|_{s-1}.\end{aligned}$$

Substituting $S_\delta \zeta' u$ for u in (2.23), we obtain

$$\begin{aligned}\|S_\delta \zeta u\|_s^2 \leqslant C\{&|(\tilde{\zeta} \Lambda^s \zeta L S_\delta \zeta' u, \tilde{\zeta} \Lambda^s \zeta S_\delta \zeta' u)| \\ &+ |(\zeta_0 \Lambda^{s-\varepsilon} \tilde{\zeta} L S_\delta \zeta' u, \zeta_0 \Lambda^{s-\varepsilon} \tilde{\zeta} S_\delta \zeta' u)| \\ &+ |(\tilde{\zeta} \Lambda^{s-\varepsilon} \zeta' L S_\delta \zeta' u, \tilde{\zeta} \Lambda^{s-\varepsilon} \zeta' S_\delta \zeta' u)| \\ &+ \|\zeta' u\|_{s-\varepsilon}^s\}.\end{aligned}$$

Proceeding as in the proof of (2.11) and using (3.2), we have

$$\begin{aligned}&|(\tilde{\zeta} \Lambda^s \zeta [L, S_\delta \zeta']\, u, \tilde{\zeta} \Lambda^s \zeta S_\delta \zeta' u)| \\ &\quad \leqslant \{\|\zeta_0 S_\delta \zeta' u\|_s^2 + \|\lambda \Lambda^s \tilde{\zeta} S_\delta \zeta' u\|^2 + \|\zeta S_\delta \zeta' u\|_s^2 + \|\zeta' u\|^2\}\end{aligned}$$

Then proceeding as in the proof of (2.23) and (2.24) we obtain (3.4) thus concluding the proof of the Main Theorem.

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